AN APPROXIMATE METHOD OF SOLVING STATIONARY HEAT CONDUCTION PROBLEMS WITH MIXED BOUNDARY CONDITIONS OF THE THIRD KIND

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We consider an approximate method of determining temperature fields when the parameter in the boundary conditions for convective heat transfer has different values on different portions of the boundary surface. We illustrate the effectiveness of our method with examples.

1. A broad class of applied problems in the theory of stationary heat conduction is connected with the need for determining temperature fields in a solid under convective heat-transfer conditions on its surface, wherein values of the heat-transfer coefficient differ, in general, from one portion of the surface to another. In this instance the corresponding boundary value problem is formulated (in nondimensional form) as follows:

$$\Delta T = 0, \tag{1}$$

$$\left.\begin{aligned} T - k_1 \frac{\partial T}{\partial n} &= C_1 |_{S_1}; \\ T - k_2 \frac{\partial T}{\partial n} &= C_2 |_{S_2}; \\ &\cdots\cdots\cdots \\ T - k_m \frac{\partial T}{\partial n} &= C_m |_{S_m}, \end{aligned}\right\} \tag{2}$$

where $k_i = 1/\mathrm{Bi}_i$ and n is along the inner normal to the boundary surface.

A rigorous solution of a problem of this kind, even in the simplest cases involving doubly-connected boundary surfaces for domains of canonical type (a halfspace, a slab, and similar configurations), turns out to be very involved and leads to a need for considering systems of integral equations (or series) (see, for example, [1, 2]). It is therefore necessary to go to an approximate analytical solution, one suitable for engineering calculations.

To construct such an approximate solution we replace the boundary conditions (2) by conditions of the form

$$\hat{T} - k_1 \frac{\partial \hat{T}}{\partial n} = f(S) \Big|_{S = \sum_{i=1}^{m} S_i}, \tag{3}$$

where

$$f(S) = \begin{cases} C_1 & \text{on } S_1, \\ C_2 + C_2' & \text{on } S_2, \\ \cdots\cdots\cdots \\ C_m + C_m' & \text{on } S_m, \end{cases} \tag{4}$$

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$$C_j' = (k_j - k_1)\frac{1}{S_j}\int_{S_j}\frac{\partial\hat{T}}{\partial n}dS, \ j = 2, 3, \ldots, m, \tag{5}$$

and $\hat{T}$ is an approximate value of the temperature.

It is readily seen that the change to boundary conditions of this kind is connected with the following assumption:

$$\left.\frac{\partial T}{\partial n}\right|_{S_j} \approx \frac{1}{S_j}\int_{S_j}\frac{\partial\hat{T}}{\partial n}dS, \tag{6}$$

here the support surface S_j, relative to which "smoothing" of the coefficients k_j ($j = 2, 3, \ldots, m$) is carried out, is taken to be the surface of largest area (or, equivalently, with the smallest value of k_j) for which there has been established, a priori, the largest nonuniformity in the distribution of the normal component of the heat flux.

The constants C'_j are obtained from the system of equations (5) after a search has been made for an approximate solution $\hat{T} = \hat{T}(k_1; C_1, C_2, \ldots, C_m; C_2', \ldots, C_m')$.

In seeking a more accurate temperature distribution on a particular portion of the boundary surface we can resort to further subdividing this portion (S_j) into yet finer portions $S_{jt}(S_j = \sum_{t=1}^{l} S_jt)$.*

We consider now an application of this method to specific examples.

2. We determine the temperature field in the halfspace $y > 0$, subject to the following boundary conditions:

$$\begin{aligned} T - k_1\frac{\partial T}{\partial y} &= 0 \quad |x| > 1, \ y = 0; \\ T - k_2\frac{\partial T}{\partial y} &= 1 \quad |x| \leqslant 1, \ y = 0. \end{aligned} \tag{7}$$

In accordance with Eqs. (3) and (4) the simplified boundary conditions for this example take on the form

$$\hat{T} - k_1\frac{\partial\hat{T}}{\partial y} = \begin{cases} 0 & |x| > 1, \ y = 0; \\ 1 + C' & |x| \leqslant 1, \ y = 0. \end{cases} \tag{8}$$

The solution of this latter problem can be easily obtained, for example, through the use of the integral Fourier transform; it has the form

$$T \approx \hat{T} = \frac{2(1 + C')}{\pi}\int_0^\infty \frac{\sin p}{(1 + k_1 p)\,p}\exp(-py)\cos px\,dp. \tag{9}$$

In accord with [3], on the boundary $y = 0$ this solution is given by the expression (after correcting the errors made in [3])

$$\begin{aligned} \hat{T} = (1 + C')\Big\{\alpha - \frac{1}{\pi}\Big[&\operatorname{ci}\frac{x+1}{k_1}\sin\frac{x+1}{k_1} - \operatorname{si}\frac{1+x}{k_1}\cos\frac{1+x}{k_1} \\ &\pm \operatorname{ci}\frac{|x-1|}{k_1}\sin\frac{|x-1|}{k_1} \mp \operatorname{si}\frac{|x-1|}{k_1}\cos\frac{|x-1|}{k_1}\Big]\Big\}, \end{aligned} \tag{10}$$

where for $|x| \leq 1$ we have $\alpha = 1$ and we take the upper sign, while for $|x| > 1$ we have $\alpha = 0$ and we take the lower sign.

At $x = 0$ (on the axis of symmetry) we have, starting from equation (9),

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* A similar approach is used, it will be recalled, in electrically modelling potential fields with boundary conditions of the third kind.

$$\hat{T}|_{x=0} = \frac{2(1+C')}{\pi}\left[\exp\left(\frac{y}{k_1}\right)\mathrm{Ei}\left(-\frac{y}{k_1}\right)\sin\frac{1}{k_1} + \frac{1}{k_1}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!\,k_1^{2n}}\sum_{l=1}^{2n}(-1)^l(l-1)!\left(\frac{k_1}{y}\right)^l\right] \tag{11}$$

and in particular,

$$\hat{T}(0,\ 0) = \frac{2(1+C')}{\pi}\left[\frac{\pi}{2} - \mathrm{ci}\,\frac{1}{k_1}\sin\frac{1}{k_1} + \mathrm{si}\,\frac{1}{k_1}\cos\frac{1}{k_1}\right]. \tag{12}$$

The unknown constant C' is determined, moreover, in accordance with Eq. (5), as

$$C' = \frac{-1}{1 + \dfrac{2}{(k_2 - k_1)I}}, \tag{13}$$

where

$$I = \frac{4}{\pi}\int_0^{\infty}\frac{\sin^2 p\,dp}{p(1+k_1 p)} = \frac{2}{\pi}\left[\ln\frac{2}{k_1} + G - \mathrm{ci}\,\frac{2}{k_1}\cos\frac{2}{k_1} - \mathrm{si}\,\frac{2}{k_1}\sin\frac{2}{k_1}\right]. \tag{14}$$

To refine this approximate solution we subdivide the strip $|x| \leq 1$ into the three portions $1 \leq x < -a$; $-a \leq x \leq a$; $a < x < 1$. We then take the boundary conditions for $|x| \leq 1$, $y = 0$ in the form

$$\hat{T} - k_1\frac{\partial \hat{T}}{\partial y}\begin{cases} 1 + C_1' & |x| \leqslant a; \\ 1 + C_2' & a < |x| \leqslant 1, \end{cases} \tag{15}$$

while the unknown coefficients are obtained from the system

$$\begin{aligned} C_1' &= -\frac{k_2 - k_1}{a}\left[I_{1a} + C_1' I_a + C_2'(I_{1a} - I_a)\right]; \\ C_2' &= -\frac{k_2 - k_1}{1-a}\left[I_1 - I_{1a} + C_1'(I_{1a} - I_a) + C_2'(I_1 + I_a - 2I_{1a})\right], \end{aligned} \tag{16}$$

where

$$I_a = \frac{1}{\pi}\left[\ln\frac{2a}{k_1} + G - \mathrm{ci}\,\frac{2a}{k_1}\cos\frac{2a}{k_1} - \mathrm{si}\,\frac{2a}{k_1}\sin\frac{2a}{k_1}\right];$$

$$I_{1a} = \frac{1}{\pi}\left[\ln\frac{1+a}{1-a} + \mathrm{ci}\,\frac{1-a}{k_1}\cos\frac{1-a}{k_1} - \mathrm{ci}\,\frac{1+a}{k_1}\cos\frac{1+a}{k_1} + \mathrm{si}\,\frac{1-a}{k_1}\sin\frac{1-a}{k_1} - \mathrm{si}\,\frac{1+a}{k_1}\sin\frac{1+a}{k_1}\right].$$

Here, $2I_1 = I$, where I is given by the expression (14). Solution of the system (16) leads to the following expressions for the unknown coefficients:

$$C_1' = R^{-1}\left[1 + \frac{k_2 - k_1}{a}(I_a - I_{1a}) + \frac{k_2 - k_1}{1-a}(I_1 + I_a - 2I_{1a})\right] - 1; \tag{17}$$

$$C_2' = R^{-1}\left[1 + \frac{k_2 - k_1}{a}I_a + \frac{k_2 - k_1}{1-a}(I_a - I_{1a})\right] - 1, \tag{18}$$

where

$$R = 1 + \frac{k_2 - k_1}{a}I_a + \frac{k_2 - k_1}{1-a}(I_1 + I_a - 2I_{1a}) + \frac{(k_2 - k_1)^2}{a(1-a)}(I_1 I_a - I_{1a}^2).$$

Comparing the solutions obtained, for example, for $k_1 = 5$, $k_2 = 10$, and $a = 0.5$, we find, after corresponding calculations, that $C' = -0.425$, while $-C_1' = 0.413$ and $-C_2' = 0.426$, whence it is easy to show that the total heat flux on the area $|X| \leq 1$, $y = 0$ is equal to $Q = 0.175$, in the first case, and to $Q = 0.170$, in the second case.

TABLE 1

k_2	Solution of the problem from its initial formulation ($k_1 = 1$)		Smoothing of parameters without an additional decomposition of the strip $\lvert x\rvert = 1$, $y = 0$			Smoothing of parameters with an additional decomposition of the strip into three parts			
	$T(0, 0)$	Q	$-C'$	$\hat{T}(0, 0)$	Q	$-C_1'$	$-C_2'$	$\hat{T}(0, 0)$	Q
1	0,6040	0,8655	—	—	—	—	—	—	—
2	0,4259	0,6027	0,3020	0,422	0,6040	0,2899	0,3137	0,4251	0,6033
5	0,2246	0,3167	0,6340	0,221	0,3168	0,6227	0,6441	0,2247	0,3167
10	0,1251	0,1770	0,7960	0,123	0,1766	0,7884	0,8022	0,1257	0,1768
100	0,0138	0,01975	0,9772	0,0138	0,01973	0,9763	0,9780	0,0140	0,01972

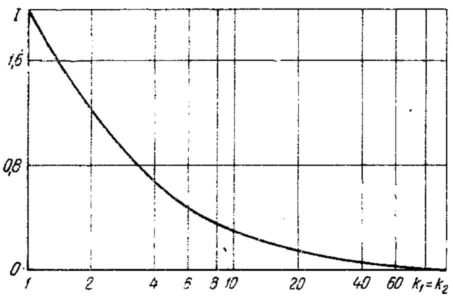

Fig. 1. Dependence of I on the parameter k_1.

Thus, the correction, which results in a more refined solution at the expense of an additional decomposition of the strip $|X| \leq 1$, $y = 0$ into individual parts, is insignificant for the relationship of the parameters k_1 and k_2 considered (amounting to less than 3%), so that for the solution of the problem we can restrict ourselves to the first approximation only.

For $k_1 = 1$ analogous calculations for various values of k_2 were also compared with the results obtained in solving the problem in its initial formulation (7) (see Table 1). This latter solution was obtained by conformal mapping of the original domain onto the interior of a disk and then applying direct methods of calculation (method of least squares). The error in the calculations, along a potential, amounted in this case to less than 5%, while along the total heat flux it amounted to less than 1%.

3. In an analogous manner the temperature field in the halfspace $z > 0$ might be found with the following boundary conditions:

$$
\begin{aligned}
T - k_1 \frac{\partial T}{\partial z} &= 0 \quad r > 1, \ z = 0; \\
T - k_2 \frac{\partial T}{\partial z} &= 1 \quad r \leqslant 1, \ z = 0.
\end{aligned}
\tag{19}
$$

An approximate solution of this problem, constructed by the same method but using the Hankel integral transform, leads to the expression

$$
\hat{T} = (1 - C') \int_0^\infty \frac{J_1(p)}{1 + k_1 p} \exp(-pz) J_0(pr)\, dp, \tag{20}
$$

where

$$
C' = -\frac{1}{1 + \dfrac{\pi}{(k_2 - k_1) I}}; \tag{21}
$$

$$
\begin{aligned}
I = 2\pi \int_0^\infty \frac{J_1^2(p)\, dp}{1 + k_1 p} = \frac{8}{k_1^2} \Bigg[\frac{2}{3}\, {}_2F_3 \left(\begin{matrix} 1, \ 1 \\ 3/2, \ 3/2, \ 5/2 \end{matrix} ; \ -\frac{1}{k_1^2} \right) \\
- {}_2F_3 \left(\begin{matrix} 1, \ 1 \\ 3/2, \ 3/2, \ 3/2 \end{matrix} ; \ -\frac{1}{k_1^2} \right) - \frac{k_1 \pi}{8} J_1\left(\frac{1}{k_1}\right) N_1\left(\frac{1}{k_1}\right) \Bigg].
\end{aligned}
\tag{22}
$$

The manner in which I varies with k_1 is shown in Fig. 1. The temperature field on the axis of the system can be obtained, in this case, from the expression (see [4])

$$
\hat{T}\big|_{r=0} = \frac{1 - C'}{k_1} \left[\exp\left(\frac{z}{k_1}\right) \mathrm{Ei}\left(-\frac{z}{k_1}\right) J_1\left(\frac{1}{k_1}\right) - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2k_1)^{2n+1} n!\,(n+1)!} \sum_{l=1}^{2n+1} (-1)^l (l-1)! \left(\frac{k_1}{z}\right)^l \right], \tag{23}
$$

and at the center of the disk, from the expression

$$\hat{T}(0,\ 0)=(1+C')\left\{1+\frac{1}{k_1}-\frac{\pi}{2k_1}\left[\mathbf{H}_1\left(\frac{1}{k_1}\right)-\mathbf{N}_1\left(\frac{1}{k_1}\right)\right]\right\}. \tag{24}$$

In conclusion, we note that the method described here can even be used in modelling the temperature field in those cases in which it is difficult to reproduce in the model, for one reason or another, the various values of the parameter k.

NOTATION

Δ, Laplace operator; Bi, Biot number; Q, total heat flux, si, ci, integral sine and cosine, respectively; Ei, integral function; J_0, J_1, Bessel functions of the zeroth and first orders; ${}_2F_3\left({\alpha_1, \alpha_2 \atop \beta_1, \beta_2, \beta_3}; -z\right)$, generalized hypergeometric series; M_1, Neuman function; H_1, Struve function; f, I, I_1, I_{1a}, I_a, R, function symbols; r, z, cylindrical coordinates; x, y, Cartesian coordinates; p, variable value; C, constant; C_j', unknown coefficients; G, Euler constant.

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